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LETTER TO THE EDITOR

Generalized coherent states as eigenstates of linearized quadratic Casimir operators

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Abstract. It is shown that the generalized coherent states for several groups and a supergroup are eigenstates of the linearized quadratic Casimir operator of the associated algebra or superalgebra. These operators may be useful in the study of geometrical phases.

Coherent states [1], and more recently vector coherent states [2], have proven to be a very powerful technique for studying the semiclassical behaviour of many quantal systems that exhibit macroscopic phenomena. Coherent states for the Heisenberg–Weyl group minimize the uncertainty relation $\Delta p \Delta x$ and are also eigenstates of the annihilation operator. In efforts to generalize the construction of coherent states to other groups, emphasis was placed on building states that minimize an appropriate uncertainty [3]. One notable exception to this was the introduction of so-called supercoherent states as eigenstates of the supersymmetrical quantum mechanical annihilation operator [4]. It turns out, however, that these states are not always minimum uncertainty states [5].

In this letter, we show that the generalized coherent states for several groups and a supergroup constructed according to the prescription given by Perelomov [3] are eigenstates of the 'linearized' quadratic Casimir operator of the algebra (or superalgebra). This operator appears to be closely related to a class of Hamiltonians recently identified in work on geometrical phases [6]. In that work, the authors studied 'linearized' Hamiltonians in the group generators and show that these are intimately related to coherent states. We now expand on what is meant by the 'linearized' quadratic Casimir operator.

Let $C_2(\mathscr{G})$ be the quadratic Casimir operator for the group, $\mathscr{G}$. Then up to an overall normalization

$$C_2(\mathscr{G}) = g^{ij} X_i X_j. \tag{1}$$

In (1), the X_k are generators of a representation of the algebra of $\mathscr{G}$ and g^{kl} is the Cartan–Killing tensor. We 'linearize' $C_2(\mathscr{G})$ by the prescription

$$A = g^{ij} \langle X_i \rangle X_j \tag{2}$$

where $\langle X_j \rangle = \langle CS | X_j | CS \rangle$ is the expectation value of X_j, and $|CS\rangle$ is a generalized coherent state constructed according to [3].

The generalized coherent states $|CS\rangle$ are eigenstates of the operator A:

$$A|CS\rangle = \lambda |CS\rangle. \tag{3}$$

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This conjecture has been considered from a different perspective for compact groups in [7]. We have not been able to formulate a general proof of this statement, but rather we wish to illustrate it with several examples. We choose as illustrations the compact group SU(2), the non-compact group SU(1, 1), and the non-compact supergroup $\mathrm{Osp}(\frac{1}{2}, R)$.

SU(2) coherent states. The compact group SU(2) has three infinitesimal generators satisfying [8]:

$$[J_+, J_-] = 2J_0 \qquad [J_0, J_\pm] = \pm J_\pm. \tag{4}$$

The quadratic Casimir operator is normalized so that [8]

$$C_2(\mathrm{SU}(2)) = J_0^2 + \tfrac{1}{2}(J_+J_- + J_-J_+) = j(j+1). \tag{5}$$

The generalized coherent state is constructed according to Perelomov [9, 10, 11]:

$$|\alpha\rangle = (1+|\alpha|^2)^{-j} \exp(\alpha J_+)|j\, -j\rangle \tag{6a}$$

with

$$\alpha = -\tan(\theta/2)\exp(-\mathrm{i}\phi). \tag{6b}$$

The expectation values of the generators of SU(2) are known to be [11]

$$\langle\alpha|J_+|\alpha\rangle = \frac{2j\alpha^*}{(1+|\alpha|^2)} \qquad \langle\alpha|J_-|\alpha\rangle = \frac{2j\alpha}{(1+|\alpha|^2)} \qquad \langle\alpha|J_0|\alpha\rangle = \frac{-j(1-|\alpha|^2)}{(1+|\sigma|^2)}. \tag{7}$$

Thus,

$$\begin{aligned} A &= \langle\alpha|J_0|\alpha\rangle J_0 + \tfrac{1}{2}\langle\alpha|J_+|\alpha\rangle J_- + \tfrac{1}{2}\langle\alpha|J_-|\alpha\rangle J_+ \\ &= -\frac{j}{(1+|\alpha|^2)}\{(1-|\alpha|^2)J_0 - \alpha J_- - \alpha J_+\} \\ &= -j\{\cos\theta\, J_0 + \sin\theta\cos\phi\, J_x + \sin\theta\sin\phi\, J_y\} = -j(\boldsymbol{n}\cdot\boldsymbol{J}). \end{aligned} \tag{8}$$

It was shown in [11] that the SU(2) coherent states defined in this manner are eigenstates of $\boldsymbol{n}\cdot\boldsymbol{J}$: $\boldsymbol{n}\cdot\boldsymbol{J}|CS\rangle = -j|CS\rangle$. Hence, we have that the SU(2) generalized coherent state is an eigenstate of the linearized quadratic Casimir operator:

$$A|CS\rangle = j^2|CS\rangle. \tag{9}$$

SU(1,1) coherent states. The non-compact group SU(1,1) has three generators satisfying the commutation relations [8]:

$$[K_+, K_-] = -2K_0 \qquad [K_0, K_\pm] = \pm K_\pm. \tag{10}$$

The quadratic Casimir operator is normalized so that [8]

$$C_2(\mathrm{SU}(1,1)) = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) = k(k-1). \tag{11}$$

We consider only the positive discrete series representations and so have a lowest weight state (LWS), $|k, k\rangle$. The generalized coherent state is then constructed as [9, 12]:

$$|\alpha\rangle = (1-|\alpha|^2)^K \exp(\alpha K_+)|k, k\rangle \tag{12a}$$

with

$$\alpha = -\tanh(\tau/2)\exp(-\mathrm{i}\psi). \tag{12b}$$

The expectation values of the generators of SU(1, 1) are given by the following [13]:

$$\langle\alpha|K_+|\alpha\rangle=\frac{2k\alpha^*}{(1-|\alpha|^2)}\qquad\langle\alpha|K_-|\alpha\rangle=\frac{2k\alpha}{(1-|\alpha|^2)}\qquad\langle\alpha|K_0|\alpha\rangle=\frac{k(1+|\alpha|^2)}{(1-|\alpha|^2)}.\tag{13}$$

Hence,

$$\begin{aligned}\exp(-\alpha K_+)A\exp(\alpha K_+)\\ &=\exp(-\alpha K_+)[\langle\alpha|K_0|\alpha\rangle K_0-(\tfrac{1}{2}\langle\alpha|K_+|\alpha\rangle K_-+\tfrac{1}{2}\langle\alpha|K_-|\alpha\rangle K_+)]\exp(\alpha K_+)\\ &=k\left(K_0-\frac{\alpha^*K_-}{(1-|\alpha|^2)}\right).\end{aligned}\tag{14}$$

It then follows that

$$A|\alpha\rangle=k(1-|\alpha|^2)^k\,e^{\alpha K_+}\left(K_0-\frac{\alpha^*K_-}{(1-|\alpha|^2)}\right)e^{-\alpha K_+}\,e^{\alpha K_+}|kk\rangle=k^2|\alpha\rangle.\tag{15}$$

The generalized coherent states for SU(1, 1) again turn out to be eigenstates of the 'linearized' quadratic Casimir operator.

Osp($\frac{1}{2}$, R) coherent states. We now consider the more complicated case of generalized coherent states for the non-compact supergroup Osp($\frac{1}{2}$, R). This non-compact supergroup has five generators satisfying [12]:

$$[K_+,K_-]=-2K_0\qquad[K_0,K_\pm]=\pm K_\pm\tag{16a}$$

$$[K_\pm,V_\pm]=0\qquad[K_\pm,V_\mp]=\mp V_\pm\qquad[K_0,V_\pm]=\pm 1/2\,V_\pm\tag{16b}$$

$$\{V_\pm,V_\pm\}=K_\pm\qquad\{V_+,V_-\}=K_0.\tag{16c}$$

The quadratic Casimir operator is normalized so that [12]

$$C(\mathrm{SU}(1,1))=K^2-\tfrac{1}{2}(V_+V_--V_-V_+)=q(q-\tfrac{1}{2}).\tag{17}$$

We again consider only the positive discrete series representations and so have a LWS, $|q,q,q\rangle$. The generalized coherent state is then constructed according to [3]

$$|\theta\alpha\rangle=(1-|\alpha|^2)^q\left(1-\frac{q\bar\theta\theta}{2(1-|\alpha|^2)}\right)\exp(\theta V_+)\exp(\alpha K_+)|q,q,q\rangle.\tag{18}$$

The expectation values of the generators of Osp($\frac{1}{2}$, R) are given by the following [14]:

$$\langle\theta\alpha|K_+|\theta\alpha\rangle=\frac{2q\alpha^*}{(1-|\alpha|^2)}\left(1-\frac{\theta\bar\theta}{2(1-|\alpha|^2)}\right)$$

$$\langle\theta\sigma|K_-|\theta\alpha\rangle=\frac{2q\alpha}{(1-|\alpha|^2)}\left(1-\frac{\theta\bar\theta}{2(1-|\alpha|^2)}\right)\tag{19a}$$

$$\langle\theta\alpha|K_0|\theta\alpha\rangle=q\frac{(1+|\alpha|^2)}{(1-|\alpha|^2)}\left(1-\frac{\theta\bar\theta}{2(1-|\alpha|^2)}\right)\tag{19b}$$

$$\langle\theta\alpha|V_+|\theta\alpha\rangle=q(\bar\theta+\alpha^*\theta)$$

$$\langle\theta\alpha|V_-|\theta\alpha\rangle=q(\theta+\alpha\bar\theta).\tag{19c}$$

Hence,

$$\exp(-\alpha K_+ - \theta V_+) A \exp(\alpha K_+ + \theta V_+)$$
$$= \exp(-\alpha K_+ + \theta V_+)\{\langle\theta\alpha|K_0|\theta\alpha\rangle K_0 - \tfrac{1}{2}[\langle\theta\alpha|K_+|\theta\alpha\rangle K_- + \langle\theta\alpha|K_-|\theta\alpha\rangle K_+]$$
$$+ \tfrac{1}{2}[\langle\theta\alpha|V_+|\theta\alpha\rangle V_- - \langle\theta\alpha|V_-|\theta\alpha\rangle V_+]\} \exp(\alpha K_+ + \theta V_+)$$
$$= q\left(1 - \frac{\theta\bar{\theta}}{2(1-|\alpha|^2)}\right)\left[K_0\left(1 - \frac{\theta\bar{\theta}}{2(1-|\alpha|^2)}\right) - \frac{\alpha^* K_-}{(1-|\alpha|^2)} + \frac{1}{2}\frac{(\bar{\theta} - \alpha^*\theta)}{(1-|\alpha|^2)} V_-\right]. \tag{20}$$

It then follows that

$$A|q\alpha\rangle = (1-|\alpha|^2)^q\left(1 - \frac{(q\bar{\theta}\theta)}{(1-|\alpha|^2)}\right) \exp(\alpha K_+ + \theta V_+) \exp(-\alpha K_+ - \theta V_+) A$$
$$\times \exp(\alpha K_+ + \theta V_+)|q, q, q\rangle$$
$$= q^2\left(1 - \frac{\theta\bar{\theta}}{(1-|\alpha|^2)}\right)|\theta\alpha\rangle. \tag{21}$$

We have illustrated that, for several groups, the generalized coherent states are eigenstates of a 'linearized' quadratic Casimir operator (2). This is similar to the random phase approximation (RPA) often used in solid state physics and nuclear physics; however, there is a fundamental difference—the results here seem to be exact.

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